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A Quantum Bousso Bound

Andrew Strominger and David Thompson

Jefferson Physical Laboratory
Harvard University
Cambridge, MA 02138

Abstract

The Bousso bound requires that one quarter the area of a closed codimension two spacelike surface exceeds the entropy flux across a certain lightsheet terminating on the surface. The bound can be violated by quantum effects such as Hawking radiation. It is proposed that at the quantum level the bound be modified by adding to the area the quantum entanglement entropy across the surface. The validity of this quantum Bousso bound is proven in a two-dimensional large N dilaton gravity theory.

1 Introduction

The generalized second law of thermodynamics (GSL) [1] roughly speaking states that one quarter the area of black hole horizons plus the entropy outside the horizons is nondecreasing. This law was formulated in an attempt to repair inconsistencies in the ordinary second law in the presence of black holes. There is no precise general statement, let alone proof, of the GSL, but it has been demonstrated in a compelling variety of special circumstances. It indicates a deep connection between geometry, thermodynamics and quantum mechanics which we have yet to fathom. The holographic principle [2, 3], which also has no precise general statement, endeavors to elevate and extend the GSL to contexts not necessarily involving black holes. In [4], a mathematically precise modification of the GSL/holographic principle was proposed that is applicable to null surfaces which are *not* horizons [5]. This proposed "Bousso bound", along with a generalization stated therein, was proven, subject to certain conditions, in a classical limit by Flanagan, Marolf, and Wald [6].

The Bousso bound, as stated, can be violated by quantum effects [7]. Mathematically, the proofs of the bound rely on the local positivity of the stress tensor which does not hold in the quantum world. Physically, the bound does not account for entropy carried by Hawking radiation. In this paper, we propose that, at a semiclassical level, the bound can be restored by adding to one quarter the surface area the entanglement entropy across the surface. We will make this statement fully precise, and then prove it, in a two-dimensional model of large N dilaton gravity.

This paper is organized as follows. We begin by reviewing Bousso's covariant entropy bound in section 2. We will review the lightsheet construction in general D-dimensional spacetime, although our main interest in the remainder of the paper will be four and two dimensions. In section 3, we will discuss how Bousso's bound can be violated in the presence of semiclassical effects, like Hawking radiation. This will motivate us to propose a "quantum Bousso bound" in section 4. By assuming an adiabaticity condition on the entropy flux, we will show in section 5 that the classical Bousso bound can be proven in four and two dimensions. In section 6, we extend the analysis to the two-dimensional RST quantization [8] of the CGHS model [9] which includes semiclassical Hawking radiation and its backreaction. We will show that the quantum Bousso bound holds in this gravitational theory.

2 Review of the classical Bousso bounds

The Bousso bound asserts that, subject to certain assumptions, the entropy of matter that passes through certain lightsheets associated with a given codimension two spatial surface in spacetime is bounded by the area of that surface [4].

This entropy bound provides a covariant recipe for associating a geometric entropy with any spatial surface B that is codimension two in the spacetime. At each point of B, there are four null directions orthogonal to B. These four null directions single out four unique null geodesics emanating from each point of B: two future-directed and two past-directed. Without loss of generality, we choose an affine parameter λ on each of these curves such that λ equals zero on B and increases positively as the geodesic is followed away from B.

Along each of the four geodesics, labelled by i, an expansion parameter $\theta_i(\lambda) = \nabla_a \left(\frac{d}{d\lambda}\right)^a$ can be defined. If we note that each of the future-directed geodesics is simply the extension of one of the past-directed geodesics, then the following relations between the expansion parameters becomes clear: $\theta_1(0) = -\theta_3(0)$, $\theta_2(0) = -\theta_4(0)$. Therefore, at least two of the four geodesics will begin with a nonpositive expansion. A "lightsheet" is a codimension one surface generated by following exactly one non-expanding geodesic from each point of B. Each geodesic is followed until one of the following occurs on it:

- The expansion parameter becomes positive, $\theta > 0$,
- A spacetime singularity is reached.

Note that, in spacetime dimensions greater than two, there are an infinite number of possible lightsheets to choose from since, for each point on B, there are at least two contracting null geodesics from which to choose.

The original Bousso bound conjectures that Nature obeys the following inequality:

$$\textbf{Entropy passing through any lightsheet of } B \; \leq \; \frac{1}{4} (\textbf{ Area of } B) \, . \tag{1}$$

In order to make this statement precise, we must clarify what we mean by the entropy that passes through a lightsheet. In general, this is ambiguous because entropy is not a local concept. However, there is a thermodynamic limit in which the entropy is well-approximated by the flux of a four-vector s^a. As discussed by Flanagan, Marolf, and Wald (FMW) in [6], this thermodynamic limit is satisfied under the entropy condition that we will use in sections 5 and 6. The Bousso bound as formulated so far pertains mainly to this limit.

To find the entropy flux that passes through the lightsheet, we must project s^a onto k^b, the unique future-directed normal to the lightsheet. Up to a sign, k is $d/d\lambda$ since $d/d\lambda$ is null and orthogonal to all other lightsheet tangent vectors by construction. In order to keep k^a future-directed, we choose $k^a = \left(\frac{d}{d\lambda}\right)^a$ if the lightsheet is future-directed, and $k^a = -\left(\frac{d}{d\lambda}\right)^a$ if the lightsheet is past-directed. Since we use the mostly-positive metric signature, the entropy flux through any point of the lightsheet is

$$s \equiv -k_a s^a \, . \tag{2}$$

In the language of entropy flux, the entropy bound becomes

$$\int_{L(B)} s \; \leq \; \frac{1}{4}(\text{Area of } B) \, , \tag{3}$$

where $L(B)$ denotes the lightsheet of B. However, there is a generalized Bousso bound [6] in which the lightsheet is prematurely terminated on a spatial surface B'. It is clear that the integral of s over this terminated lightsheet equals the integral over the full lightsheet of B minus the integral over the full lightsheet of B'. Assuming that s is everywhere positive, Bousso's original entropy bound tells us that

$$\int_{L(B-B')} s \;\leq\; \int_{L(B)} s \;\leq\; \frac{1}{4}(\text{Area of } B)\,, \tag{4}$$

where $L(B-B')$ denotes the lightsheet of B terminated on B'. In this paper, we will be interested in the generalized Bousso bound, first proposed by FMW [6], which imposes the much stronger bound on the terminated lightsheet:

$$\int_{L(B-B')} s \;\leq\; \frac{1}{4}\left(A(B) - A(B')\right)\,. \tag{5}$$

This has been proven under suitable assumptions by FMW [6]. Note that this generalized entropy bound directly implies Bousso's original entropy bound.

3 Semiclassical violations

The entropy bounds so far pertain largely to the classical regime. When quantum effects are included, even at the semiclassical level, we expect that the bounds must be somehow modified to account for the entropy carried by Hawking radiation. Mathematically, the proofs [6] are not applicable because quantum effects violate the positive energy condition.

The classical proofs hinge on the focussing theorem of classical general relativity. The focussing theorem, in turn, derives from the Raychaudhuri equation and the null energy condition. The Raychaudhuri equation provides a differential equation for the expansion parameter along a null geodesic [10]:

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 8\pi T_{ab}k^a k^b\,, \tag{6}$$

where σ_{ab} is the shear tensor and ω_{ab} is the twist tensor. For a family of null geodesics that start off orthogonal to a spatial surface, such as the case for a lightsheet, the twist tensor is zero. Finally, if we assume that the null energy condition holds, then the last term is negative. The null energy condition postulates that $T_{ab}k^a k^b$ is nonnegative for all null vectors k^a. As a result, we find that the expansion parameter satisfies the inequality

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{D-2}\theta^2\,. \tag{7}$$

This gives us the focussing theorem: If the expansion parameter takes the negative value θ_0 along a null geodesic of the lightsheet, then that geodesic will reach a caustic (i.e., $\theta \to -\infty$) within the finite affine time $\Delta\lambda \leq \frac{D-2}{|\theta_0|}$.

So long as energy is required to produce entropy, the focussing theorem ensures that the presence of entropy will cause the lightsheet to reach a caustic and, therefore, terminate. The more entropy we try to pass through the lightsheet, the faster the lightsheet terminates. This gives a compelling argument for why only a finite, bounded amount of entropy could be passed through the lightsheet. According to the Bousso bound, this upper bound is precisely one quarter the area of the generating surface.

In practice, the covariant entropy bound can be violated in the presence of matter with negative energy. By mixing positive-energy matter and negative-energy matter, a system with zero energy can be made to carry an arbitrary amount of entropy. Again, the entropy passing through any given lightsheet could be increased arbitrarily. At the classical level, we could simply demand that the energy-momentum tensor obeys the null energy energy condition. This is the weakest of all the most common energy conditions and, as can be seen from (6), is the one needed for the focussing theorem, and thus to make the Bousso bound plausible.

However, the Bousso bound is in serious trouble once we include quantum effects. We know that none of the local energy conditions can hold even at first order in $\hbar$. In particular, the phenomenon of Hawking radiation violates the null energy condition near the horizon of black holes. This allows for violations of the focussing theorem. This violation can be seen most clearly for future-directed, outgoing null geodesics that hover for a while in between the event horizon and apparent horizon of an evaporating black hole. The apparent horizon is the boundary of the region of trapped surfaces, so the congruence of null geodesics are contracting inside the apparent horizon. However, as the black hole evaporates, the apparent horizon follows a timelike trajectory towards the event horizon. The null geodesic could then leave the apparent horizon and begin expanding, in violation of the focussing theorem.

Furthermore, in [7], Lowe constructs a related counterexample to the covariant entropy bound in the presence of a critically illuminated black hole. Critical illumination is the process in which matter is thrown into a black hole at exactly the same rate as energy is Hawking radiated away. In this scenario, the apparent horizon follows a null trajectory. If we pick the apparent horizon to be the generating surface for a lightsheet, then the lightsheet will coincide with the apparent horizon as long as we continue to critically illuminate the black hole. By critically illuminating the black hole sufficiently long, we can pass an arbitrary amount of matter through the lightsheet. In this way, the entropy of the matter passing through the lightsheet can be made larger than the area of the apparent horizon, thus violating the entropy bound.

Hence, the original Bousso bound only has a chance of holding in the classical regime. Once we include one-loop quantum effects, such as Hawking radiation, the bound fails. In the remaining sections, we propose a modification of the Bousso bound which may hold in the semiclassical regime.

4 A quantum Bousso bound

The generalized Bousso bound, when specialized to black hole horizons, is equivalent to a classical limit of the generalized second law of thermodynamics (GSL). To see this, note that the portion of the event horizon lying between any two times consitutes a lightsheet. Since all matter falling into the black hole between those two times must pass through this lightsheet, the generalized entropy bound gives us the same information as the GSL. In particular, we learn that

$$\frac{1}{4}\Delta A_{\mathrm{EH}} \geq \Delta S_{\mathrm{m}}\,, \tag{8}$$

where ΔA_{EH} is the change in event horizon area, and ΔS_{m} is the entropy of the matter that fell in.

When quantum effects are included, the form (8) of the generalized second law is no longer valid. The quantum GSL states, roughly speaking, that the total entropy outside the black hole plus one quarter the area of the horizon (either event or apparent depending on the formulation) is non-decreasing. The entropy outside the black hole receives an important contribution from Hawking radiation. Therefore, we must augment the left hand side by the entropy of the Hawking radiation:

$$\frac{1}{4}\Delta A_{\mathrm{H}} + \Delta S_{\mathrm{hr}} \geq \Delta S_{\mathrm{m}} \tag{9}$$

In general, we do not know how to formulate, let alone prove, an exact form of the GSL in a full quantum theory of gravity. However, approximations to it have been formulated and demonstrated in a wide variety of circumstances [11]. The ΔS_{hr} term is crucial in these demonstrations, without which counterexamples may be easily found.

Since the GSL requires an additional term at the quantum level, and the GSL is a special case of the generalized Buosso bound, we should certainly expect that the Bousso bound will receive related quantum corrections. These corrections should reduce to ΔS_{hr} when the lightsheets are taken to be portions of event horizons. The problem is to precisely formulate the nature of these corrections.

In this context it is useful to think of the entropy in Hawking radiation as entanglement entropy. Evolution of the quantum fields on a fixed black hole geometry is a manifestly unitary process prior to singularity formation. Nevertheless entropy is created outside the black hole because the outgoing Hawking quanta are correlated with those that fall behind the horizon. When a region of space U is unobservable, we should trace the quantum state ψ over the modes in the unobservable region to obtain the observable density matrix ρ,

$$\rho = \mathrm{tr}_U \,|\psi\rangle\langle\psi|\,. \tag{10}$$

Since the full state is in principle not available to the observer, there is a de facto loss of information that can be characterized by the entanglement entropy

$$S_{\mathrm{ent}} = -\mathrm{tr}\,\rho \log \rho\,. \tag{11}$$

In general, this expression has divergences and requires further definition, which will be given below for the case of two dimensions.[1] Choosing U to be the region behind the horizon, we can therefore formally identify

$$\Delta S_{\rm hr} = \Delta S_{\rm ent}. \tag{12}$$

This motivates a natural guess for quantum corrections to the Bousso bound when the initial and final surfaces are closed. One should add to the area the entanglement entropy across the surface. Applying this modification to the classical Bousso bound (5) results in a quantum Bousso bound of the form:

$$\int_{L(B-B')} s \ \leq \ \frac{1}{4}A(B) + S_{\rm ent}(B) - \frac{1}{4}A(B') - S_{\rm ent}(B')\,. \tag{13}$$

Since we can not presently hope to solve this problem or even define this quantum bound in exact quantum gravity, in order to go further we need to identify a small expansion parameter for approximating the exact theory. A useful parameter, which systematically captures the quantum corrections of Hawking radiation, is provided by $\frac{1}{N}$, where N is the number of matter fields and $G_N N$ is held fixed [9]. In [13] it was shown in the two-dimensional RST model of black hole evaporation that the (suitably defined) GSL, incorporating the Hawking radiation as in (9), is valid. One might hope that a similar incorporation can save the Bousso bound.

In the process of the investigations in [13] it emerged that the sum $A + 4S_{\rm ent} \equiv A_{\rm qu}$ arises naturally in the theory as a kind of quantum-corrected area. In this paper, we propose that the required leading $\frac{1}{N}$ semiclassical correction to the generalized Bousso bound simply involves the replacement of the classical area with this quantum corrected area. A precise version of this statement will be formulated and proved in the RST model in section 6.

5 Proving the classical Bousso bounds

In this section we reproduce proofs of classical Bousso bounds. We first give a proof due to Bousso, Flanagan, and Marolf of the generalized Bousso bound in four dimensions [14].[2] This simplified proof follows from conditions on the initial entropy flux and an adiabaticity condition on the rate of change of the entropy flux which differ somewhat from the conditions assumed in [6]. We then describe a two dimensional version of the proof obtained by spherical reduction. A small modification of this gives a proof of the generalized Bousso bound in the classical CGHS model [9], which is then transcribed into Kruskal gauge for later convenience. The inclusion of quantum effects in the latter will be the subject of the next section.

[1] UV divergences in this expression are absorbed by the renormalization of Newton's constant [12].

[2] We thank Raphael Bousso and Eanna Flanagan for explaining this proof prior to publication.

5.1 Simplified proof in four dimensions

Following [6], the integral of the entropy flux s over the lightsheet can be written as

$$\int_{L(B-B')} s \;=\; \int_B d^2x\sqrt{h(x)}\int_0^1 d\lambda\, s(x,\lambda)\mathcal{A}(x,\lambda)\,. \tag{14}$$

In this expression, we have chosen a coordinate system (x^1, x^2) on the spatial surface B, $h(x)$ is the determinant of the induced metric on B, and the affine parameter on each null geodesic of the lightsheet has been normalized so that $\lambda = 1$ is when the geodesic reaches B'. The function $\mathcal{A}(x,\lambda)$ is the area decrease factor for the geodesic that begins at the point x on B. In terms of θ, it is given by

$$\mathcal{A} \equiv \exp\left[\int_0^\lambda d\tilde{\lambda}\,\theta(\tilde{\lambda})\right]\,. \tag{15}$$

The physical intuition for equation (14) is simple. As we parallel propagate a small coordinate patch of area $d^2x\sqrt{h(x)}$ from the point $(x,0)$ on B to the point (x,λ) on the lightsheet, the area contracts to $d^2x\sqrt{h(x)}\mathcal{A}(x,\lambda)$. The proper three-dimensional volume of an infinitesimal cube of the lightsheet is $d^2x\,d\lambda\sqrt{h(x)}\mathcal{A}(x,\lambda)$, and this volume times $s(x,\lambda)$ gives the entropy flux passing through that cube. In order to prove the generalized entropy bound, it is sufficient to prove that

$$\int_0^1 d\lambda\, s(\lambda)\,\mathcal{A}(\lambda) \le \frac{1}{4}(1-\mathcal{A}(1)) \tag{16}$$

for each of the geodesics that comprise the lightsheet.

Using a mostly positive metric signature, the assumed entropy conditions are

i. $s' \le 2\pi\, T_{ab}k^a k^b$

ii. $s(0) \le -\frac{1}{4}\,\mathcal{A}'(0)$,

where we use the notation, both here and henceforth, that primes denote differentiation with respect to the affine parameter λ. Condition **i** is very similar to one of the conditions in [6]. It can be interpreted as the requirement that the rate of change of the entropy flux is less than the energy flux, which is a necessary condition for the thermodynamic approximation to hold. Condition **ii** requires only that the covariant entropy bound is not violated infinitesimally at the beginning of the lightsheet. Since the square root of $\mathcal{A}$ routinely appears in calculations, we borrow the notation of FMW and define

$$G \equiv \sqrt{\mathcal{A}}\,. \tag{17}$$

From the Raychaudhuri equation, we have that

$$T_{ab}k^a k^b = -\frac{1}{4\pi}\frac{G''(\lambda)}{G(\lambda)} - \frac{1}{8\pi}\sigma_{ab}\sigma^{ab} \le -\frac{1}{4\pi}\frac{G''(\lambda)}{G(\lambda)}\,, \tag{18}$$

where σ_{ab} is the shear tensor, and the inequality follows from the fact that $\sigma_{ab}\sigma^{ab} \geq 0$ always. Now we see that

$$\begin{aligned}
s(\lambda) &= \int_0^\lambda d\tilde{\lambda}\, s'(\tilde{\lambda}) + s(0) \\
(i) \qquad &\leq 2\pi \int_0^\lambda d\tilde{\lambda} T_{ab} k^a k^b + s(0) \\
(eom) \qquad &\leq 2\pi \int_0^\lambda d\tilde{\lambda} \left(-\frac{1}{4\pi} \frac{G''(\tilde{\lambda})}{G(\tilde{\lambda})} \right) + s(0) \\
&= \frac{1}{2} \left(\frac{G'(0)}{G(0)} - \frac{G'(\lambda)}{G(\lambda)} \right) - \frac{1}{2} \int_0^\lambda d\tilde{\lambda} \frac{G'(\tilde{\lambda})^2}{G(\tilde{\lambda})^2} + s(0) \\
(ii) \qquad &\leq -\frac{1}{2} \frac{G'(\lambda)}{G(\lambda)} - \frac{1}{2} \int_0^\lambda d\tilde{\lambda} \frac{G'(\tilde{\lambda})^2}{G(\tilde{\lambda})^2} \\
&\leq -\frac{1}{2} \frac{G'(\lambda)}{G(\lambda)} .
\end{aligned}$$

Consequently,

$$\int_0^1 d\lambda\, s(\lambda) G(\lambda)^2 \leq -\frac{1}{2} \int_0^1 d\lambda\, G(\lambda) G'(\lambda) = \frac{1}{4} \left(\mathcal{A}(0) - \mathcal{A}(1) \right) . \tag{19}$$

We have shown that, given our entropy conditions, the entropy passing through a lightsheet is bounded by one quarter the difference in area of the two bounding spatial surfaces. This is precisely the statement of the generalized Bousso bound.

It is interesting to note that nowhere in the proof did we need to use the contracting lightsheet condition. The only indication that we should choose a contracting lightsheet comes from the boundary condition **ii**. We see from condition **ii** that, in order to allow a positive, future-directed entropy flux, the derivative of $\mathcal{A}$ must be negative. If the lightsheet were expanding at $\lambda = 0$, then a timelike entropy flux would have to be past-directed at $\lambda = 0$.

Note also that Bousso's entropy bound can be saturated only if $G' = 0$ for all λ. In light of the Raychaudhuri equation (18), we see that T_{ab} and the shear σ_{ab} must be zero everywhere along the lightsheet in order for G' to remain zero. The bound can be saturated only in this most trivial scenario. This will not be the case for other gravitational theories we will study, such as the CGHS dilaton model, where saturation of the bound can occur in the presence of matter.

5.2 Spherical reduction

Our goal is to study the entropy bound in two dimensional models where our semiclassical analysis will be greatly simplified. As a guide to what phenomenological conditions we should be using in 2D models, we will first rederive the previous proof for the purely spherical sector of 4D Einstein-Hilbert gravity.

We begin with the 4D Einstein-Hilbert action coupled to some matter Lagrangian density, $\mathcal{L}_m$:

$$\int d^4x\sqrt{-g^{(4)}}\left(\frac{R^{(4)}}{16\pi}+\mathcal{L}_m^{(4)}\right) . \tag{20}$$

Writing the four-dimensional metric as

$$(ds^2)^{(4)} = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{-2\phi(x)}(d\vartheta^2+\sin^2\vartheta\, d\varphi^2) \quad \mu,\nu\in\{0,1\}\,, \tag{21}$$

and integrating over the angular coordinates we find the action is reduced to

$$\int d^2x\sqrt{-g}\left[e^{-2\phi}\left(\frac{1}{4}R+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}e^{2\phi}\right)+\mathcal{L}_m\right] . \tag{22}$$

Here, the 2D matter Lagrangian density $\mathcal{L}_m$ is related to $\mathcal{L}_m^{(4)}$ by

$$\mathcal{L}_m = 4\pi e^{-2\phi}\mathcal{L}_m^{(4)} . \tag{23}$$

From the equations of motion, we conclude that

$$k^a k^b T_{ab} = -e^{-\phi}k^a k^b\nabla_a\nabla_b e^{-\phi}\,, \tag{24}$$

whenever k is a null vector. In this expression, T is the energy-momentum tensor for $\mathcal{L}_m$, not $\mathcal{L}_m^{(4)}$.

It is clear from the four-dimensional metric that the classical "area" of a point in the 2D model is $A_{\rm cl} = 4\pi e^{-2\phi}$. However, had we only been given the action, we could identify the "area" of a point as being proportional to the factor multiplying the Ricci scalar in the Lagrange density. If that were not convincing enough, we could study the thermodynamics of a black hole solution of the two-dimensional model. In particular, we would first determine the mass of a stationary black hole solution and then compute the temperature of the Hawking radiation on this geometry (neglecting backreaction). Integrating the thermodynamic identity $dS = dM/T$ and identifying S as one-quarter the area of the event horizon, we arrive at an expression for the area of the event horizon in terms of the local values of the various fields there. We then designate this function of local fields as the expression that gives us the "area" of any point in the two-dimensional space.

Deriving the two-dimensional entropy conditions is a simple matter of rewriting the four-dimensional conditions in terms of two-dimensional tensors. For example, we replace $T_{ab}^{(4)}$ with $\frac{1}{4\pi}e^{2\phi}T_{ab}$. We are also interested in the two-dimensional entropy flux s_a which is related to the four-dimensional entropy flux $s_a^{(4)}$ by $s_a^{(4)} = \frac{1}{4\pi}e^{2\phi}s_a$. This relation is a simple consequence of the fact that the 2D flux at a point equals the 4D flux up to an overall factor of the area of the corresponding S^2. Replacing 4D tensors with 2D tensors, we arrive at the following entropy conditions:

i. $e^{-2\phi}(s\,e^{2\phi})' \leq 2\pi T_{ab}\,k^a k^b$

ii. $s(0) \leq -\frac{1}{4}A'_{\rm cl}(0)$

Note that we continue to use $s \equiv -k^a s_a$ and primes denoting $d/d\lambda$. Putting it all together, the derivation of the entropy bound goes through in the same way as it did in the 4D case. In detail, we find

$$
\begin{aligned}
s(\lambda) &= e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(s(\tilde{\lambda})e^{2\phi(\tilde{\lambda})}\right)' + e^{-2\phi(\lambda)}s(0)e^{2\phi(0)} \\
(i) \quad &\leq e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda}\, 2\pi\, e^{2\phi(\tilde{\lambda})} k^a k^b T_{ab}(\tilde{\lambda}) + e^{-2\phi(\lambda)}s(0)e^{2\phi(0)} \\
(eom) \quad &= -2\pi e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(e^{-\phi(\tilde{\lambda})}\right)'' e^{\phi(\tilde{\lambda})} + e^{-2\phi(\lambda)}s(0)e^{2\phi(0)} \\
&= -2\pi e^{-2\phi(\lambda)} \left(-\phi'(\lambda) + \phi'(0)\right) - 2\pi e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(\phi'(\tilde{\lambda})\right)^2 + e^{-2\phi(\lambda)}s(0)e^{2\phi(0)} \\
(ii) \quad &\leq -\pi \left(e^{-2\phi(\lambda)}\right)' - 2\pi e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(\phi'(\tilde{\lambda})\right)^2 \\
&\leq -\pi \left(e^{-2\phi(\lambda)}\right)'
\end{aligned}
$$

Therefore,

$$
\int_0^1 d\lambda\, s(\lambda) \leq \pi \left(e^{-2\phi(0)} - e^{-2\phi(1)}\right) = \frac{1}{4}\left(A_{\rm cl}(0) - A_{\rm cl}(1)\right)\,, \tag{25}
$$

which is exactly the 4D entropy bound, only derived from the 2D perspective.

5.3 CGHS

Although we have derived an entropy bound in a 2D model using 2D entropy conditions, we were guaranteed success since we had spherically reduced a successful 4D proof. We now attempt to apply the same entropy conditions to another 2D dilaton gravity model, namely the CGHS model [9]. The CGHS model can also be derived as the spherical reduction of a 4D model, but with charges. In what follows, we will work purely at the 2D level without any recourse to higher-dimensional physics. The CGHS action coupled to N conformal matter fields with Lagrangian density $\mathcal{L}_m$ is

$$
\int d^2x\sqrt{-g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2 + 4) + \mathcal{L}_m\right]\,. \tag{26}
$$

For a null vector k^a, the equations of motion give

$$
k^a k^b T_{ab} = -2e^{-\phi} k^a k^b \nabla_a \nabla_b e^{-\phi} + 2k^a k^b \nabla_a e^{-\phi} \nabla_b e^{-\phi}\,. \tag{27}
$$

To determine the classical "area" of a point, we look at the coefficient of the Ricci scalar and learn that it is proportional to $e^{-2\phi}$. By studying black hole thermodynamics, the constant of proportionality can be fixed as $A_{\rm cl} = 8e^{-2\phi}$.

To prove the entropy bound, we start with the following assumptions:

i. $e^{-2\phi}(s\, e^{2\phi})' \leq 2\, T_{ab}\, k^a k^b$

ii. $s(0) \leq -\frac{1}{4} A'_{\rm cl}(0)$

We will continue to use $s \equiv -k^a s_a$ and primes denoting $d/d\lambda$. Putting it all together, we find

$$\begin{aligned}
s(\lambda) &= e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(s(\tilde{\lambda})\, e^{2\phi(\tilde{\lambda})} \right)' + e^{-2\phi(\lambda)} s(0) e^{2\phi(0)} \\
(i) \quad &\leq 2\, e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda}\, e^{2\phi(\tilde{\lambda})} k^a k^b T_{ab}(\tilde{\lambda}) + e^{-2\phi(\lambda)} s(0) e^{2\phi(0)} \\
(eom) \quad &= -4\, e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(e^{-\phi(\tilde{\lambda})} \right)'' e^{\phi(\tilde{\lambda})} + 4\, e^{-2\phi(\lambda)} \int_0^\lambda d\tilde{\lambda} \left(\phi'(\tilde{\lambda}) \right)^2 + e^{-2\phi(\lambda)} s(0) e^{2\phi(0)} \\
&= -4\, e^{-2\phi(\lambda)} \left(-\phi'(\lambda) + \phi'(0) \right) + e^{-2\phi(\lambda)} s(0) e^{2\phi(0)} \\
(ii) \quad &\leq -2 \left(e^{-2\phi(\lambda)} \right)' .
\end{aligned}$$

Therefore, we find the desired relation:

$$\int_0^1 d\lambda\, s(\lambda) \leq 2 \left(e^{-2\phi(0)} - e^{-2\phi(1)} \right) = \frac{1}{4} \left(A_{\rm cl}(0) - A_{\rm cl}(1) \right) . \tag{28}$$

5.4 CGHS in Kruskal gauge

In the previous section, we derived the CGHS entropy bound with manifestly covariant equations of motion and entropy conditions. However, once we add the one-loop trace anomaly, we are only able to obtain local equations of motion in conformal gauge. Furthermore, our calculations are greatly simplified in Kruskal gauge. Therefore, it behooves us to rederive the CGHS result in Kruskal gauge.

We will assume that the lightsheet moves in the decreasing x^+ direction. Our results for this past-directed x^+ lightsheet generalize simply to the other three possible lightsheet directions. Working with the x^+ lightsheet, we will be interested in the following equation of motion:

$$T_{++} = -2e^{-\phi} \nabla_+ \nabla_+ e^{-\phi} + 2 \nabla_+ e^{-\phi} \nabla_+ e^{-\phi} . \tag{29}$$

In conformal gauge, the RHS can be written as $2e^{-2\phi} \left(\partial_+ \partial_+ \phi - 2 \partial_+ \rho \partial_+ \phi \right)$. In Kruskal gauge, we set $\rho = \phi$, so this becomes

$$T_{++} = -\partial_+ \partial_+ e^{-2\phi} . \tag{30}$$

Since $k^+ = -\partial x^+/\partial\lambda = e^{-2\phi}$ in Kruskal gauge, our entropy conditions can be rewritten in Kruskal gauge coordinates as

i. $\partial_+ s_+ \le 2\,T_{++}$

ii. $-s_+(x_0^+) \le \frac{1}{4}\,\partial_+ A_{\rm cl}(x_0^+)$.

Recall that $s \equiv -k^+ s_+$, so $-s_+$ is positive so long as the proper entropy flux s is positive.

Applying these conditions, we find

$$\begin{aligned}
-s_+(x^+) &= \int_{x^+}^{x_0^+} d\tilde{x}^+ \partial_+ s_+(\tilde{x}^+) - s_+(x_0^+) \\
(i) \qquad &\le 2\int_{x^+}^{x_0^+} d\tilde{x}^+ T_{++}(\tilde{x}^+) - s_+(x_0^+) \\
(eom) \qquad &= 2\,\partial_+ e^{-2\phi}\Big]_{x_0^+}^{x^+} - s_+(x_0^+) \\
(ii) \qquad &\le 2\,\partial_+ e^{-2\phi(x^+)}\,.
\end{aligned}$$

We find that

$$\int_0^1 d\lambda\, s(\lambda) = \int_{x_0^+}^{x_1^+} d\tilde{x}^+ s_+(\tilde{x}^+) = \int_{x_1^+}^{x_0^+} d\tilde{x}^+ \left(-s_+(\tilde{x}^+)\right) \le \frac{1}{4}\left(A_{\rm cl}(0) - A_{\rm cl}(1)\right)\,. \tag{31}$$

Had we chosen the future-directed x^+ lightsheet, then we would have $k^+ = \partial x^+/\partial\lambda = e^{-2\phi}$ and our entropy conditions would have been $\partial_+(-s_+) \le 2\,T_{++}$ and $-s_+(x_0^+) \le -\frac{1}{4}\,\partial_+ A_{\rm cl}(x_0^+)$. The extension to x^- lightsheets is trivial.

6 Stating and proving a quantum Bousso bound

The classical CGHS action is

$$S_{\rm CGHS} = \int d^2x\sqrt{-g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2 + 4) + \mathcal{L}_m\right]\,. \tag{32}$$

For N conformal matter fields, Hawking radiation and its backreaction on the geometry can be accounted for by adding to the classical CGHS action the nonlocal term

$$S_{\rm PL} = -\frac{N}{48}\int d^2x\sqrt{-g(x)}\int d^2x'\sqrt{-g(x')}R(x)G(x,x')R(x')\,, \tag{33}$$

where $G(x,x')$ is the Green's function for ∇^2. This is a one loop quantum correction but it nevertheless contributes at leading order in a $\frac{1}{N}$ expansion. At the one loop level, there is the freedom of also adding a local counterterm to the action. The large N theory becomes analytically soluble if we add the following judiciously chosen local counterterm to the action [8]:

$$S_{\text{ct}} = -\frac{N}{24}\int d^2x\sqrt{-g}\phi R\,. \tag{34}$$

The full action for the RST model is then

$$S_{\text{RST}} = S_{\text{CGHS}} + S_{\text{PL}} + S_{\text{ct}}\,. \tag{35}$$

We can once again choose Kruskal gauge, but this time $\rho = \phi + \frac{1}{2}\log(N/12)$. In conformal and Kruskal gauges, the equations of motion become

$$\partial_+\partial_-\Omega = -1\,, \tag{36}$$

and

$$\partial_\pm^2\Omega = -\frac{12}{N}T_{\pm\pm} - t_\pm\,, \tag{37}$$

where

$$\Omega = \frac{12}{N}e^{-2\phi} + \frac{\phi}{2} + \frac{1}{4}\log\frac{N}{48}\,. \tag{38}$$

The $t_\pm$ term in (37) accounts for the normal-ordering ambiguity. We wish to consider semi-classical excitations built on the vacuum state which has no positive frequency excitations according to inertial observers on $\mathcal{I}^-$. These inertial coordinates, $\sigma^\pm$, are related to the Kruskal coordinates by

$$x^+ = e^{\sigma^+}\,, \quad x^- = -e^{-\sigma^-}\,. \tag{39}$$

For coherent states built on this σ vacuum $t_\pm = 0$ in σ coordinates. Its value in Kruskal coordinates is given by the Schwarzian transformation law as

$$t_\pm = -\frac{1}{4(x^\pm)^2}\,. \tag{40}$$

As worked out in [13], the classical "area" of a point in the RST model is

$$A_{\text{cl}} = 8e^{-2\phi} - \frac{N}{3}\phi - \frac{N}{6} - \frac{N}{6}\log\frac{N}{48}\,. \tag{41}$$

For coherent states built on the σ vacuum, the renormalized entanglement entropy across a point has a local contribution

$$S_{\text{ent}} = \frac{N}{6}\left(\phi + \frac{1}{2}\log\frac{N}{12} + \frac{1}{2}\log\left(-x^+x^-\right)\right)\,. \tag{42}$$

The full entanglement entropy also has an infrared term which is not locally associated to the horizon and so is not included here. See [13] for a detailed derivation and discussion of these points.

Now, Ω can be written as

$$\Omega = \frac{3}{2N}\left(A_{\rm cl} + 4S_{\rm ent}\right) - \frac{1}{2}\log(-x^+x^-) - \log 2 + \frac{1}{4}\,. \tag{43}$$

Differentiating, we obtain

$$\partial_+\Omega + \frac{1}{2x^+} = \frac{3}{2N}\partial_+ A_{\rm qu}\,. \tag{44}$$

When analyzing the RST model, we will leave entropy condition **i** unchanged. In the formulation of **ii**, we will replace $A_{\rm cl}$ with $A_{\rm qu} \equiv A_{\rm cl} + 4S_{\rm ent}$ In Kruskal coordinates, these become

i. $\partial_+ s_+ \le 2\,T_{++}$

ii. $-s_+(x_0^+) \le \frac{1}{4}\partial_+ A_{\rm qu}(x_0^+)$

$$\begin{array}{rcl}
-s_+(x^+) & = & \displaystyle\int_{x^+}^{x_0^+} d\tilde{x}^+ \partial_+ s_+(\tilde{x}^+) - s_+(x_0^+) \\
(i) & \le & \displaystyle 2\int_{x^+}^{x_0^+} d\tilde{x}^+ T_{++}(\tilde{x}^+) - s_+(x_0^+) \\
(eom) & = & \displaystyle \frac{N}{6}\left(\partial_+\Omega + \frac{1}{4x^+}\right)\Bigg]_{x_0^+}^{x^+} - s_+(x_0^+) \\
& = & \displaystyle \left(\frac{1}{4}\partial_+ A_{\rm qu} - \frac{N}{24x^+}\right)\Bigg]_{x_0^+}^{x^+} - s_+(x_0^+) \\
(ii) & \le & \displaystyle \frac{1}{4}\partial_+ A_{\rm qu}(x^+) - \frac{N}{24}\frac{1}{x^+} + \frac{N}{24}\frac{1}{x_0^+} \\
& \le & \displaystyle \frac{1}{4}\partial_+ A_{\rm qu}(x^+)\,.
\end{array}$$

We find

$$\int_0^1 d\lambda\, s(\lambda) = \int_{x_0^+}^{x_1^+} d\tilde{x}^+ s_+(\tilde{x}^+) = \int_{x_1^+}^{x_0^+} d\tilde{x}^+\left(-s_+(\tilde{x}^+)\right) \le \frac{1}{4}\left(A_{\rm qu}(0) - A_{\rm qu}(1)\right)\,. \tag{45}$$

With our entropy conditions, we see that the covariant entropy bound is satisfied once we replace $A_{\rm cl}$ with $A_{\rm qu}$.

It is interesting to note that the quantum Bousso bound can not be saturated for coherent states built on the σ vacuum. The obstruction to saturation is the term $\frac{N}{24}\left(\frac{1}{x_0^+} - \frac{1}{x^+}\right)$ that shows up in the calculation of $-s_+(x^+)$. However, had we built our state on top of the Kruskal vacuum (i.e., the Hartle-Hawking state), we would have $t_+ = 0$ and $S_{\text{ent}} = \frac{N}{6}(\phi + \frac{1}{2}\log\frac{N}{12})$. As a result, both our equations of motion and our definition of A_{qu} would change in a way that eliminates the $\frac{N}{24}\left(\frac{1}{x_0^+} - \frac{1}{x^+}\right)$ term from the calculations. The quantum Bousso bound will then be saturated any time the two entropy conditions are saturated.

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